The Hadamard gate cannot be replaced by a resource state in universal quantum computation

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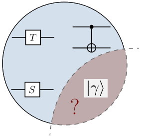

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- We give evidence that whilst you can siphon off the resource of magic and entanglement in a supplementary state, this does not hold for coherence: i.e. some coherence must be present in the operations.
- Our technical results relate to the resource theory of coherence.

Overview

1. Motivation
2. Results
3. Outlook

Table of Contents

Clifford + T

- Clifford + T gate set.

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Hadamard

Controlled-NOT

Phase

T

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

$$CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix},$$

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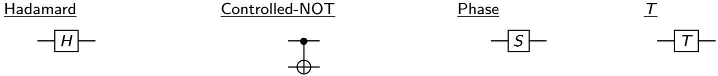

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- H, $CNOT$, T respectively provide coherence, entanglement, magic.

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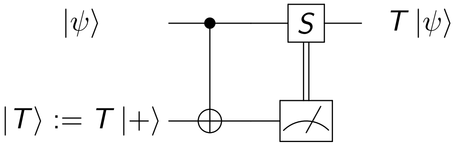

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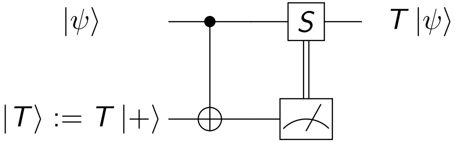

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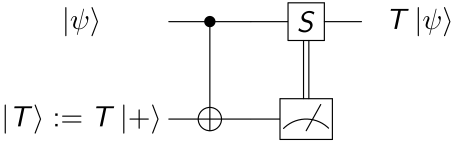

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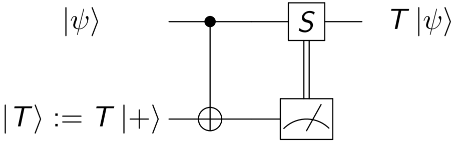

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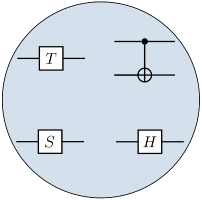

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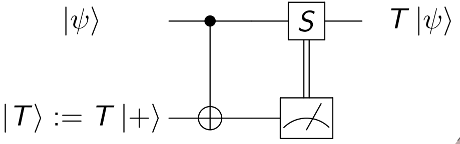

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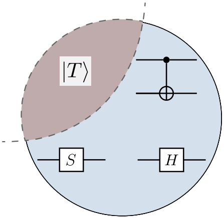

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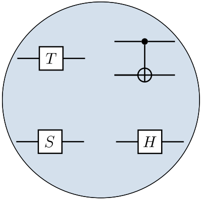

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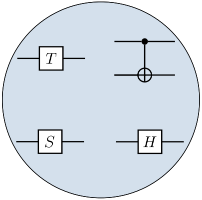

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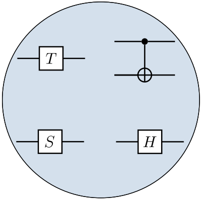

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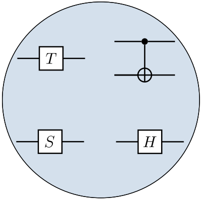

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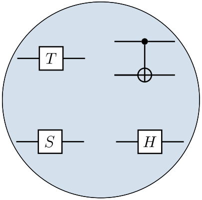

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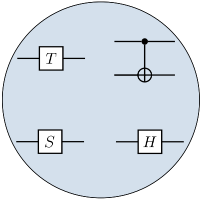

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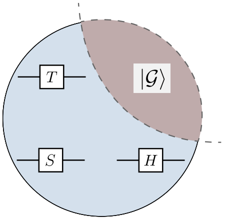

Free operations acting on resourceful state.

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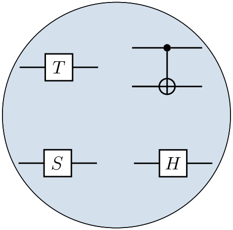

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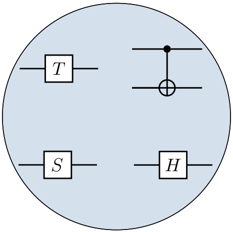

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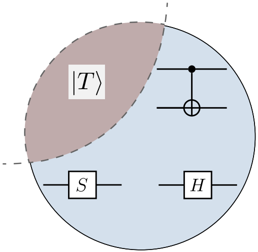

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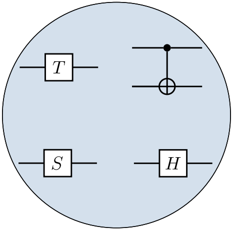

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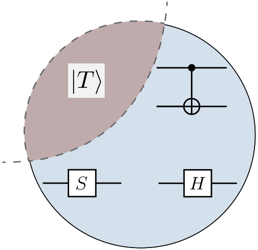
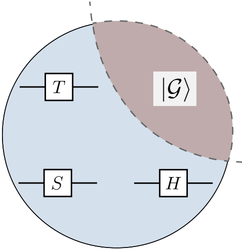


(c) MBQC.



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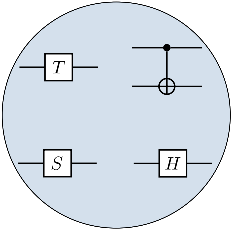

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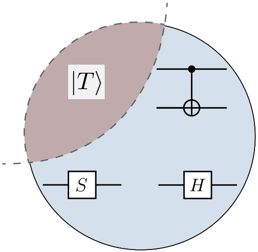

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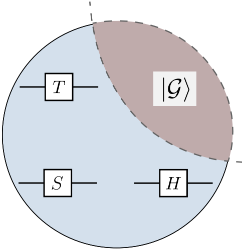

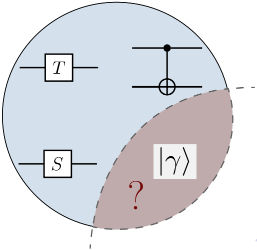

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Motivating Questions

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- Can we find a unified framework for quantum computation using free operations on resourceful states?
- Is there an implementation of the Hadamard gate using only CNOT, S and T gates, classical control, computational basis measurements and an ancillary state?
- More generally, where can we put the 'cut' between gates and states and still achieve universal quantum computation? We can put all the magic and entanglement in a supplementary state, but what about coherence?

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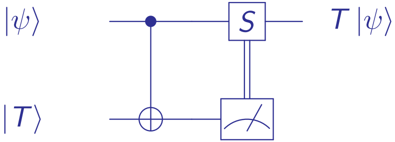
First ideas: a Hadamard gadget?



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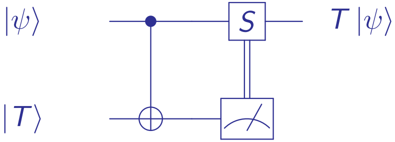

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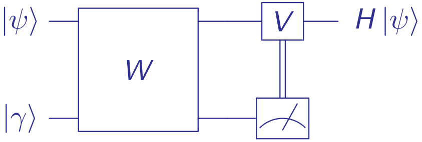

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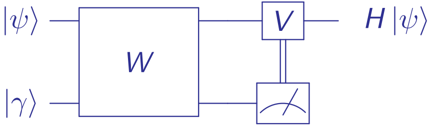

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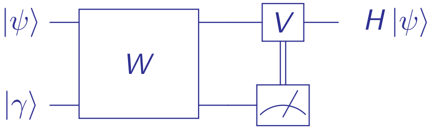

We can replace classical control with quantum control:

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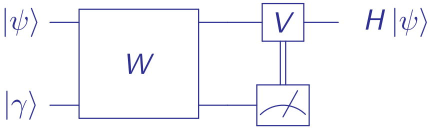

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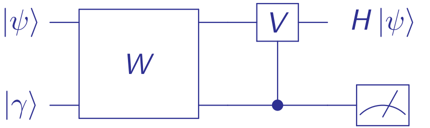

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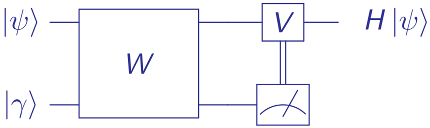

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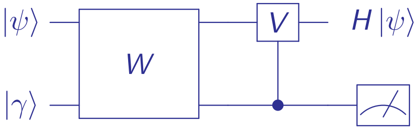

Deterministic implementation corresponds to partial trace:

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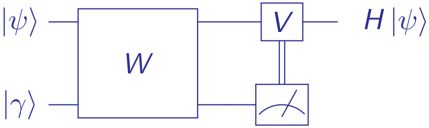

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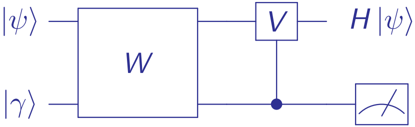

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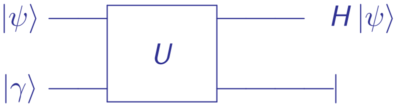

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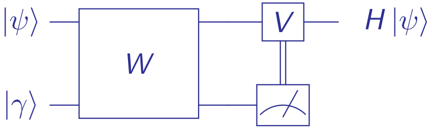

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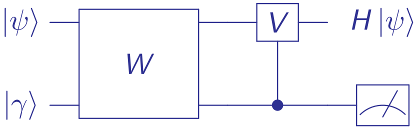

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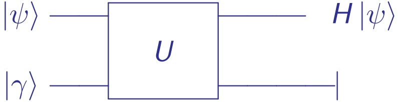
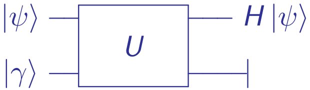


Does there exist a product of (possibly controlled) gates of $CNOT$, T, S for U?

First ideas



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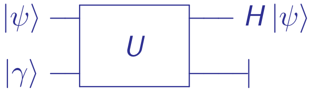

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- CNOT, S and T are all instances of *incoherent unitaries*, i.e. are of the following form for some permutation π and real numbers θ_x.

$$U = \sum_{x=1}^{d} e^{i\theta_x} |\pi(x)\rangle\langle x| \tag{5}$$

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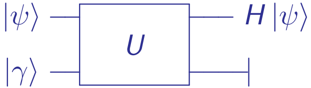

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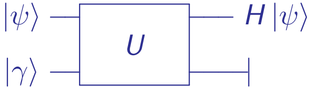

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We ask generally if there exists a (controlled) incoherent U and arbitrary state $|\gamma\rangle$ s.t.

$$\mathcal{E}(\rho) = \text{Tr}_2 \left(U\rho \otimes |\gamma\rangle\langle\gamma| \, U^\dagger \right) = H\rho H^\dagger \qquad \forall \rho. \tag{6}$$

Some Preliminaries

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- Given some set of unitaries $\mathcal{U}$ and a preferred basis $\{|x\rangle\}$, we denote by $\mathcal{C}(\mathcal{U})$ the corresponding set of generalised controlled unitaries:

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Most general channel possible to implement deterministically can be written as

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This incorporates lots of examples: MSI, MBQC, matchgates, Pauli-based QC ...
In our case, $\mathcal{U}$ are incoherent unitaries, so $\mathcal{C}(\mathcal{U})$ are also incoherent, and as SWAP is incoherent we can WLOG take the trace over the second system.

Result 1

$$\mathcal{E}(\rho) = \mathrm{Tr}_2\left(U\rho \otimes |\gamma\rangle\langle\gamma|\, U^\dagger\right) \stackrel{?}{=} H\rho H^\dagger \qquad \forall\rho \tag{9}$$

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Theorem

Given the ability to perform incoherent unitaries, computational basis measurements and classical control, it is impossible to implement any coherent unitary (e.g. Hadamard) exactly, even when supplemented with an arbitrary ancilla.

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Theorem

Given the ability to perform incoherent unitaries, computational basis measurements and classical control, it is impossible to implement a single Hadamard to within induced trace distance of $\frac{1}{4}$.

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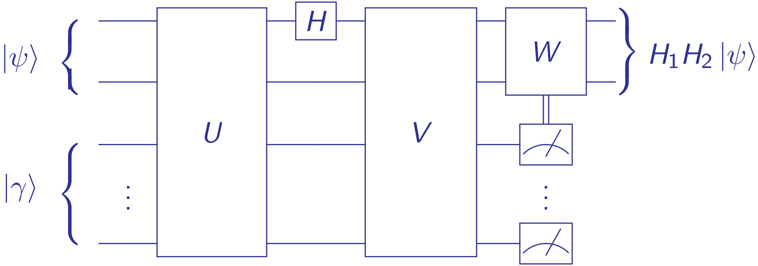

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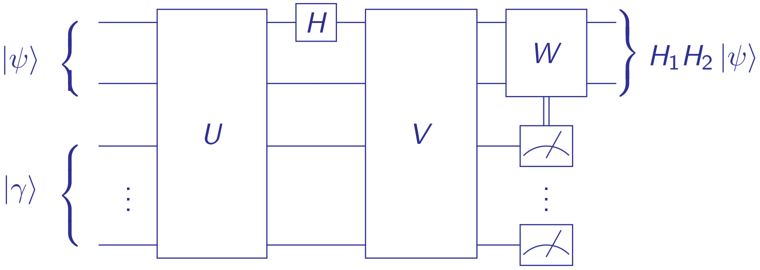
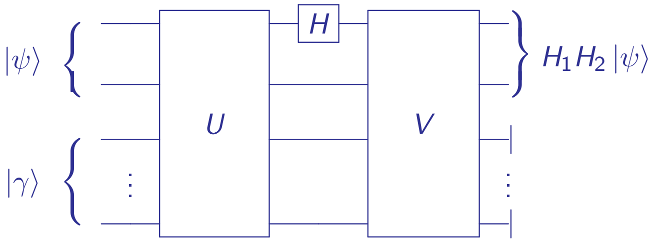


Again we could write this as . . .

Extension to k Hadamards



$$VH_1 U |\psi\rangle |\gamma\rangle = H_1 H_2 |\psi\rangle |\gamma_\psi\rangle \quad (12)$$

For incoherent U and V, and some $|\gamma_\psi\rangle$ that could a priori depend on $|\psi\rangle$.

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- We now need to introduce the coherence rank.

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which is a contradiction (as $r \geq 1$).

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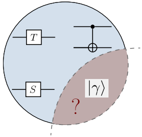

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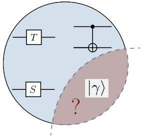

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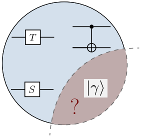

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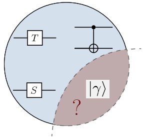

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- Our proofs went via the resource theory of coherence.

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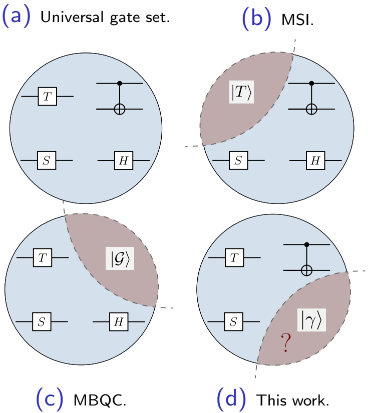

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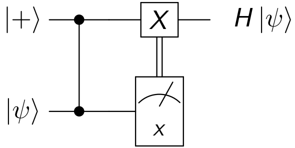
arXiv:2312.03515

Hadamard Gadgets

So-called Hadamard gadgets are known to exist, e.g.:



(23)

However, they crucially rely on X basis measurements, that is, measurements in the coherent basis $\{|+\rangle, |-\rangle\}$. We show that such gadgets cannot exist if one restricts to computational basis measurements.